243. Montagsgespräch

„Symmetrien“

Dr. Jutta Köhler

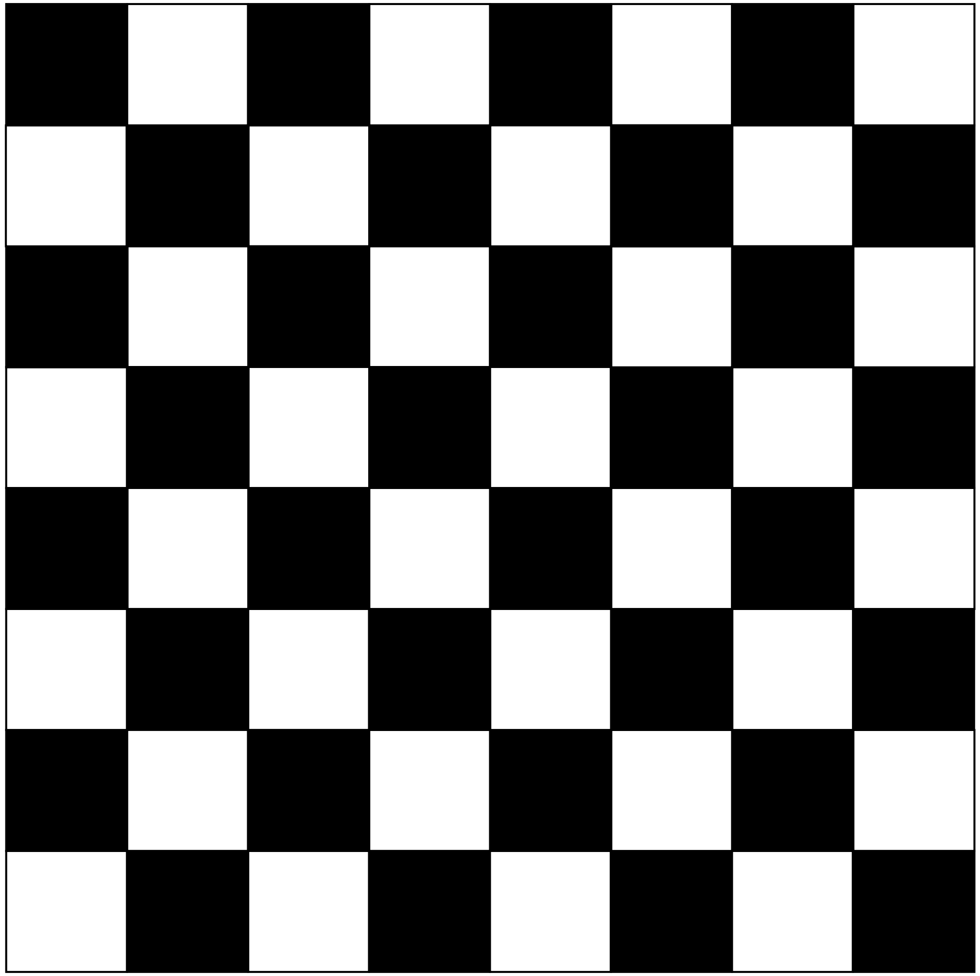
am 12. März 2007

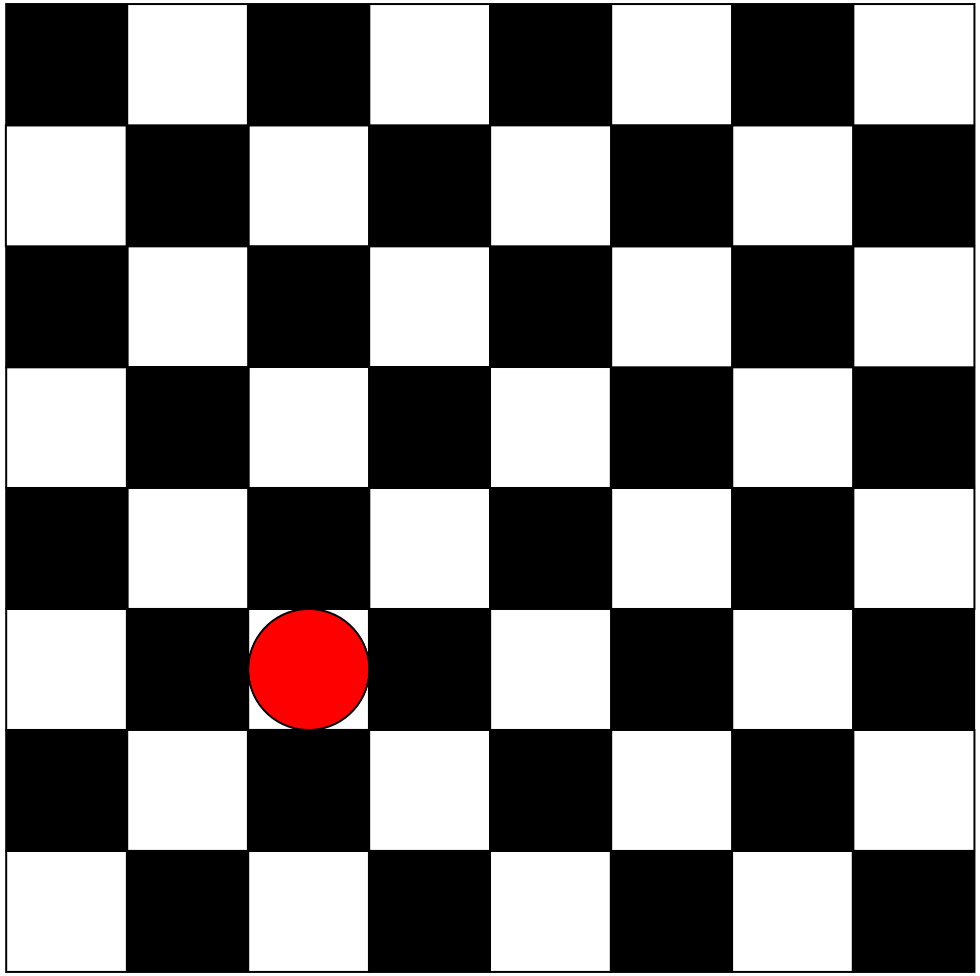
im Musiklabor der Hochschule für Musik und Theater München

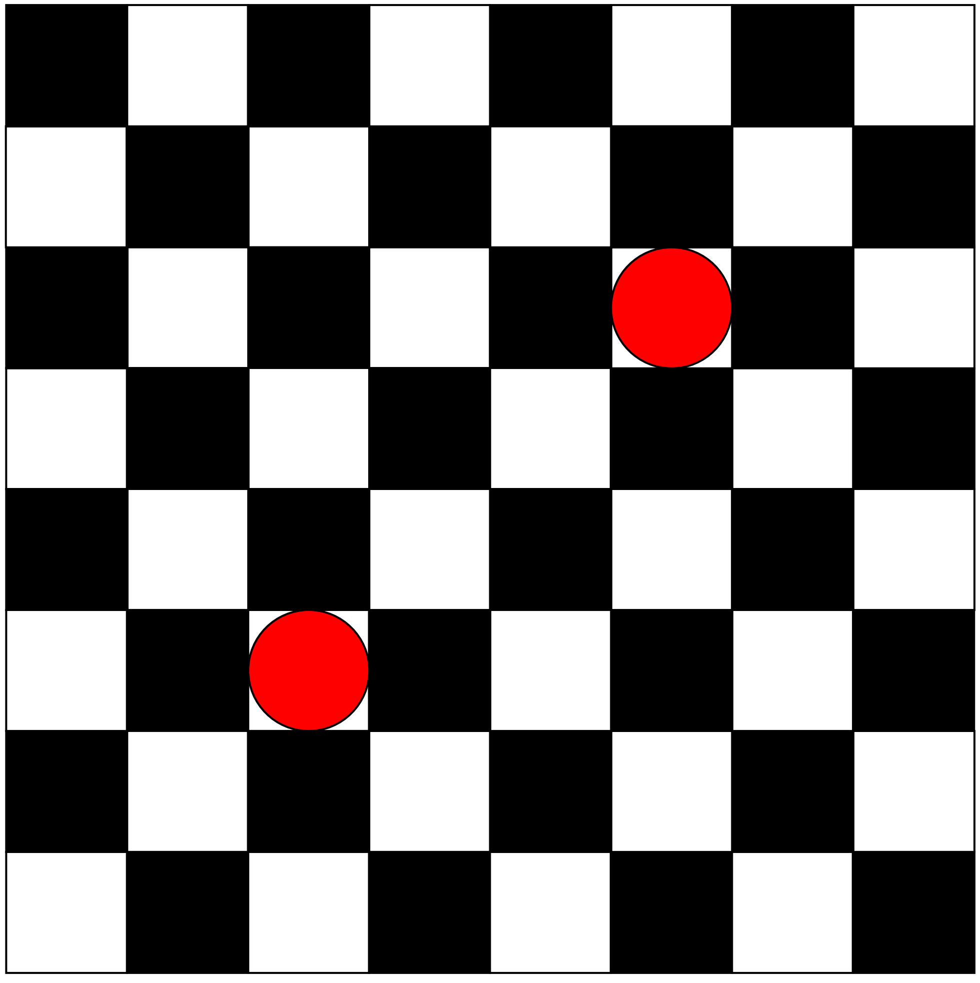
Carl Orff Auditorium

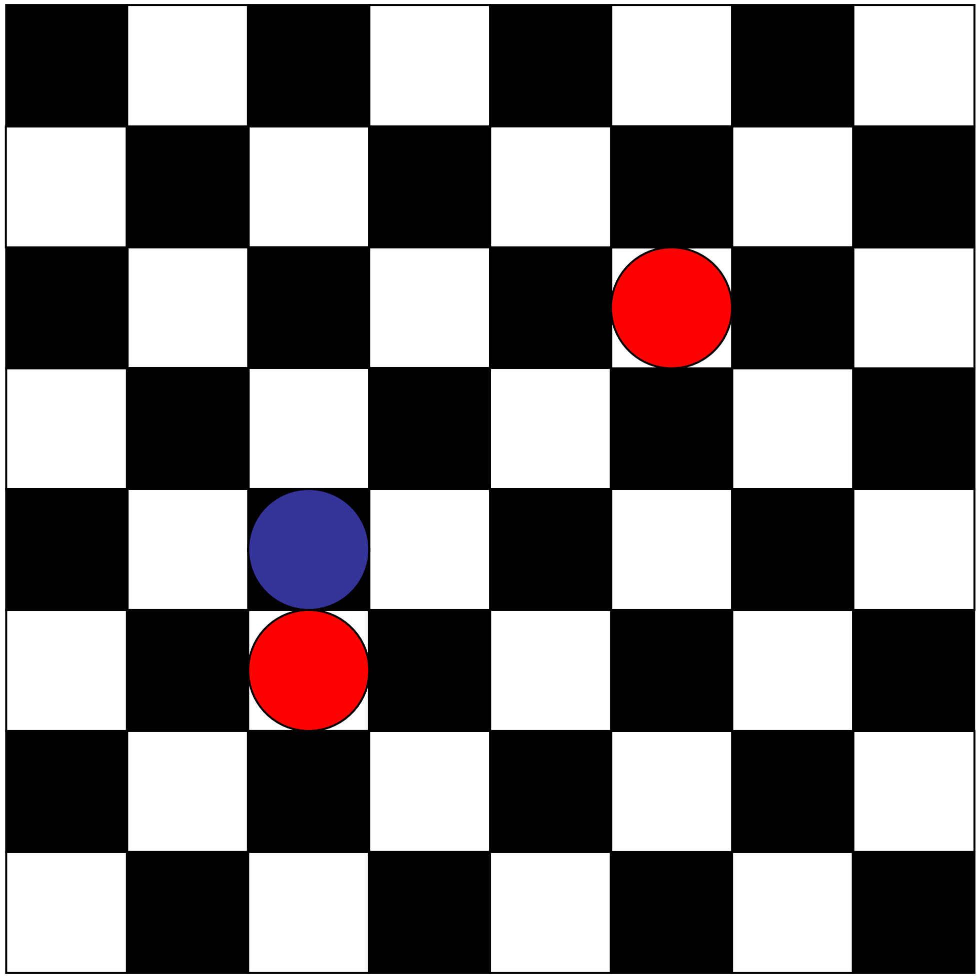
Luisenstrasse 37a, München

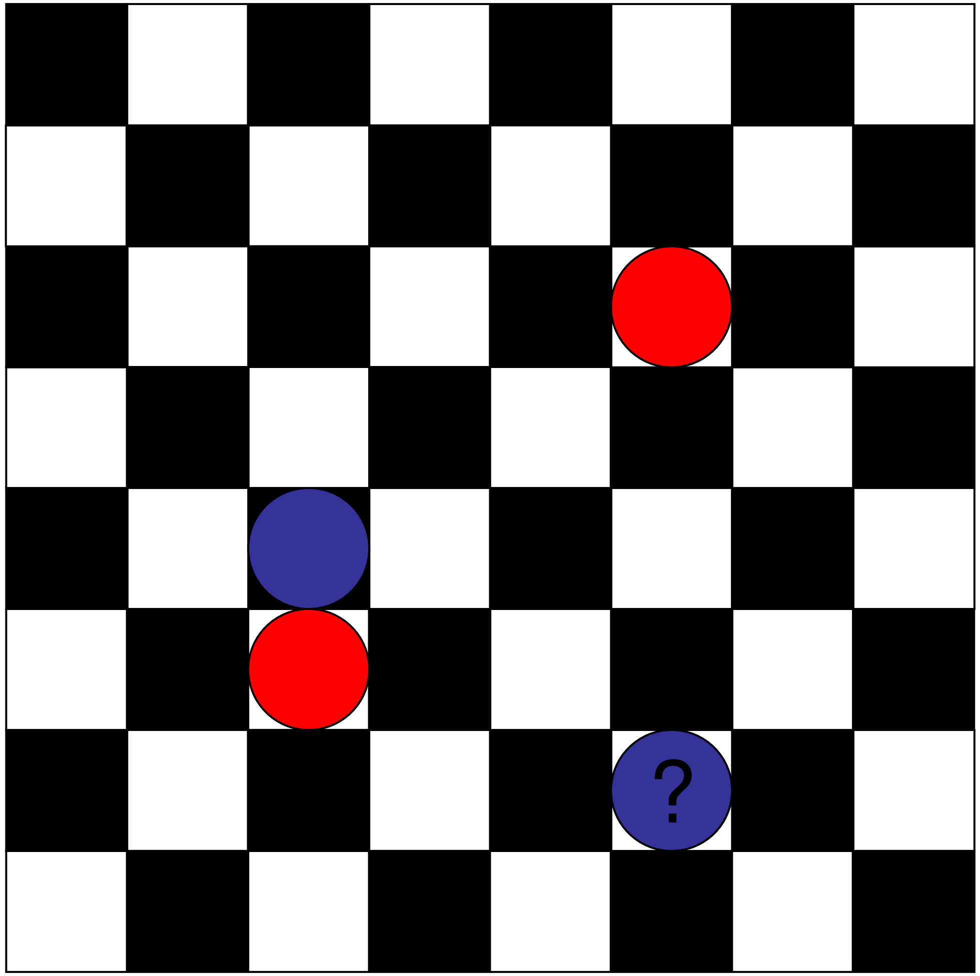
Die offensichtlichen und die verborgenen Symmetrien im Schachbrettmuster











Die verborgene Symmetrie der Sprache

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..e .i.. i. .o. a .e..e. .o .e .i..e. .u. a
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e i i o a e e o e i e u a
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The mind is not a vessel to be filled but a fire to be kindled.

Plutarch

Der Geist ist kein Gefäß zum Füllen sondern ein Feuer zum entfachen.

The mind is not a vessel to be filled but a fire to be kindled.

Plutarch

Der Geist ist kein Gefäß zum Füllen sondern ein Feuer zum entfachen.

The more progress physical sciences make, the more they tend to enter the domain of mathematics, which is a kind of centre to which they all converge.

We may even judge the degree of perfection to which a science has arrived by the facility with which it may be submitted to calculation.

Adolphe Quetelet 1796-1874

Tymoczko in Princeton with piano and orbifold

PETER MURPHY FOR TIME

Die Geometrie der musikalischen Akkorde

Dmitri Tymoczko, Princeton Universität

Musikalische Akkorde haben eine nicht Euklidische Geometrie, welche von westlichen Komponisten in vielen verschiedenen Stilen genutzt wurde.

Ein musikalischer Akkord kann als ein Punkt in einem geometrischen Raum, welcher „orbifold“ genannt wird, dargestellt werden. Liniensegmente repräsentieren Abbildungen der Noten eines Akkords auf die eines anderen. Komponisten verschiedenster Stilrichtungen haben die nicht-Euklidische Geometrie dieses Raumes verwendet, typischerweise durch kurze Liniensegmente zwischen strukturell ähnlichen Akkorden. Solche Liniensegmente existieren nur, wenn die Akkorde fast symmetrisch sind in Bezug auf Translation, Reflexion oder Permutation. Paradigmatisch konsonante oder dissonante Akkorde besitzen unterschiedliche „near-Symmetrien“ und sie haben unterschiedliche musikalische Bedeutungen.

SYMBOL OR TERM	DEFINITION
multiset, object, member, element	A multiset is an unordered collection in which duplications are permitted. {0, 1, 1} is the same multiset as {1, 0, 1} but is different from {0, 1}. The multiset {0, 1, 1} contains three objects and has three members. However, it has only two elements, 0 and 1.
$\{a, b, c\}$	The multiset with members a, b, c, some of which may be identical.
(a, b, c)	An ordered list. (a, b, c) and (b, c, a) are distinct.
$i \in \boldsymbol{I}$	Object i belongs to set or multiset $\boldsymbol{I}$.
group	A group is a set whose elements can be combined so as to satisfy certain axioms. See a group theory textbook for details.
$\mathbb{R}$	The real numbers.
$\mathbb{R}^n$	The set of ordered n-tuples $(x_1, x_2, \ldots x_n)$ such that each $x_i \in \mathbb{R}$.
$\mathbb{Z}$	The integers.
$n\mathbb{Z}$, where $n \in \mathbb{R}$	The set $\{nk \mid k \in \mathbb{Z}\}$. Thus $12\mathbb{Z}$ is the set $\{\ldots, -24, -12, 0, 12, 24, \ldots\}$. The elements of this set form a group under addition.
$m\mathbb{Z}^n$, where $m \in \mathbb{R}$ and $n \in \mathbb{Z}$	The set of ordered n-tuples $(x_1, x_2, \ldots x_n)$ such that each $x_i \in m\mathbb{Z}$. These n-tuples form a group under vector addition.
$\mathcal{S}_n$	The symmetric group of degree n, consisting of the group of permutations of n objects.
quotient space	A quotient space is formed by identifying (or "gluing together") points in another space.
$\boldsymbol{A}/\mathcal{G}$, where $\mathcal{G}$ is some group of transformations acting on $\boldsymbol{A}$	The quotient space that identifies all points a and ga, where $a \in \boldsymbol{A}$ and $g \in \mathcal{G}$.
$\mathbb{R}/12\mathbb{Z}$	The circular quotient space in which all real numbers x and $x + 12$ are identified. Points in this space are infinite sets of the form $\{\ldots, x - 24, x - 12, x, x + 12, x + 24, \ldots\}$, where $x \in \mathbb{R}$. These sets can be labeled using real numbers in the range $0 \le x < 12$. The elements of $\mathbb{R}/12\mathbb{Z}$ form a group under addition of their labels modulo $12\mathbb{Z}$.

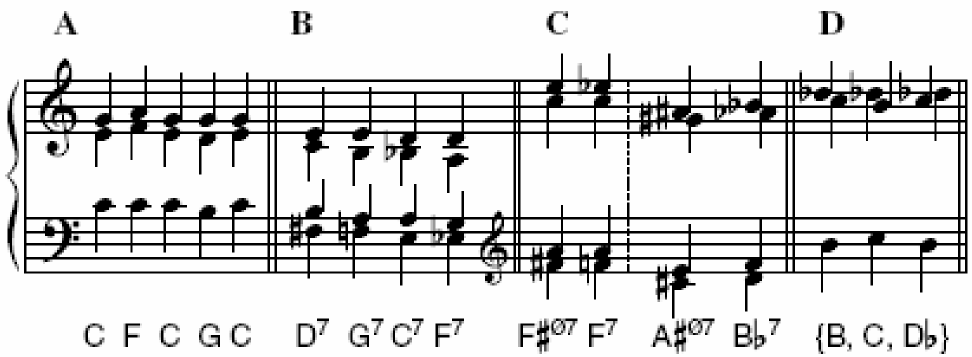


Fig. 1. Efficient voice leading between transpositionally and inversionally related chords. These progressions exploit three near-symmetries: transposition (A-B), inversion (C), and permutation (D). Sources: classical music (A), jazz (B), Wagner's *Parsifal* (C), Debussy's *Faun* (C), and contemporary atonality (D) (Soundfile S1).

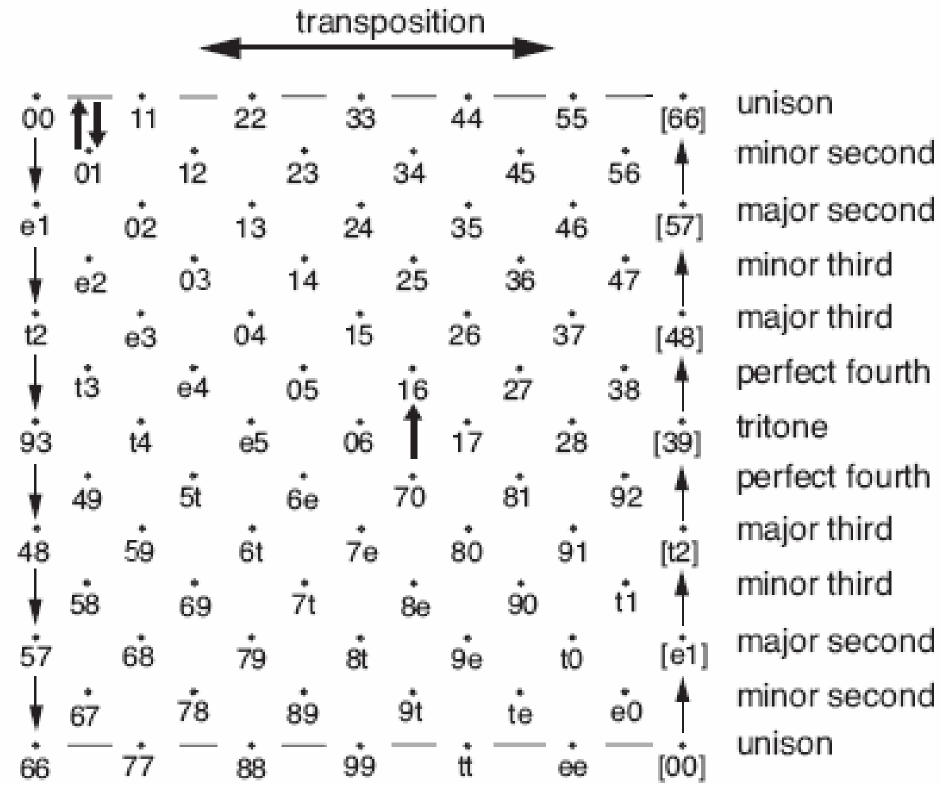


Fig. 2. The orbifold $\mathbb{T}^2/\mathcal{S}_2$. C = 0, C♯ = 1, etc., with B♭ = t, and B = e. The left edge is identified with the right. The voice leadings (C, D♭)→(D♭, C) and (C, G)→(C♯, F♯) are shown; the first reflects off the singular boundary.

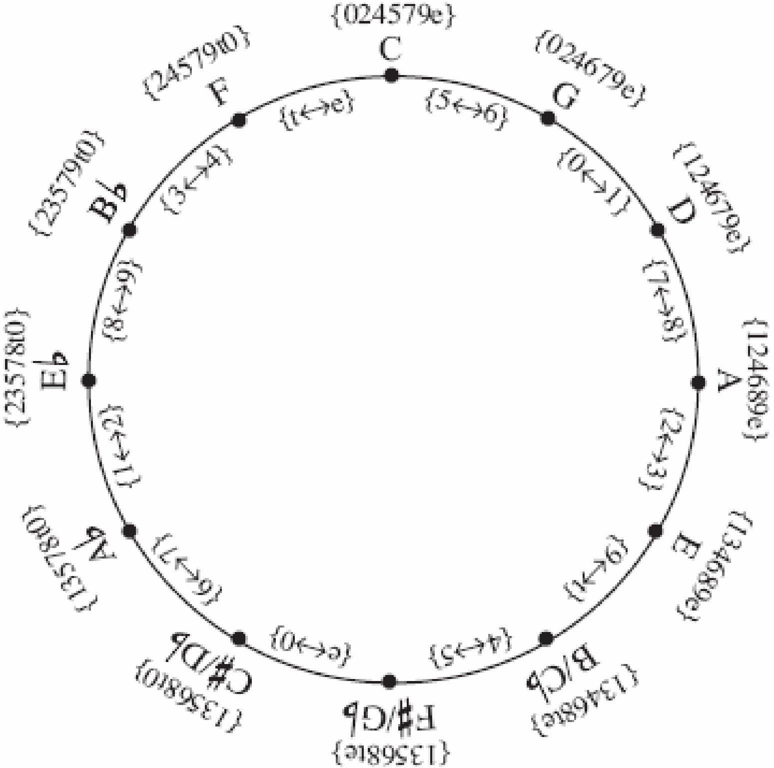


Figure S1. The circle of fifths depicts minimal voice leadings between diatonic collections (major scales). Each diatonic collection can be transformed into its neighbors by moving one pitch class by one semitone. For example, the C major scale can be transformed into the G major scale by moving the pitch class 5 (F) to 6 (F♯). Here and elsewhere, the letters “t” and “e” refer to 10 (B♭) and 11 (B), respectively.

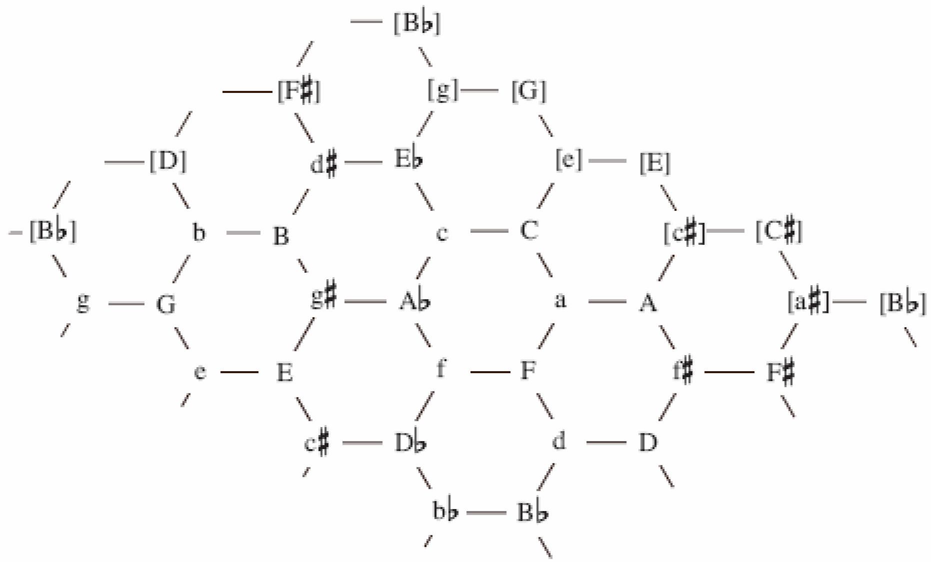


Figure S2. The Tonnetz. Nineteenth-century theorists such as Hostinsky, Oettingen, and Riemann explored a graph that is the geometric dual of the one shown here. The graph displays efficient voice leadings among the 24 familiar major and minor triads. Uppercase and lowercase letters indicate major and minor triads, respectively. Triads connected by horizontal lines share both root and fifth, and can be linked by voice leading in which one note moves by one semitone. (For example, the C major triad can be transformed into a C minor triad by changing E to E♭.) Triads connected by a NE/SW diagonal also share two notes and can be linked by single-semitone voice leading. (For example, the C major triad can be transformed into an E minor triad by changing C to B.) Triads connected by a NW/SE diagonal share two notes and can be linked by voice leading in which one note moves by two semitones. (For example, the C major triad can be transformed into an A minor triad by changing G to A.)

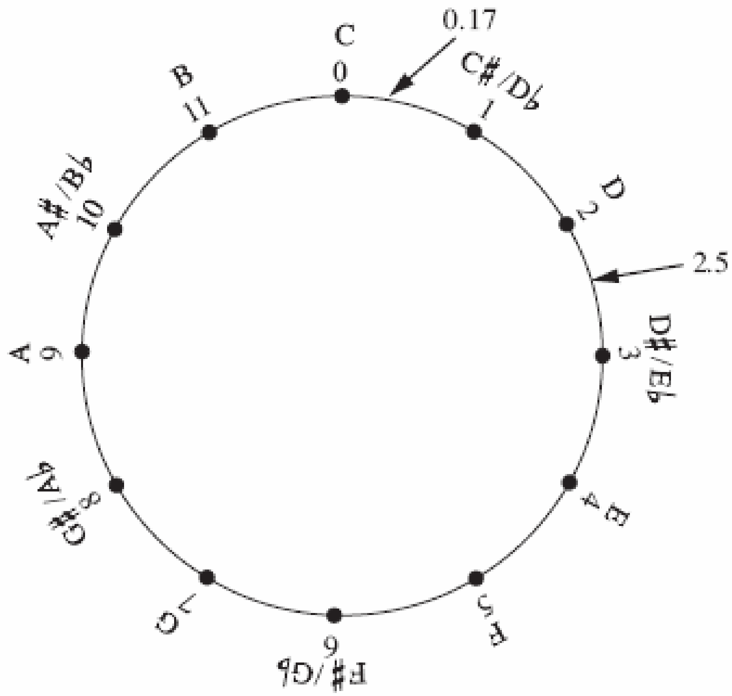
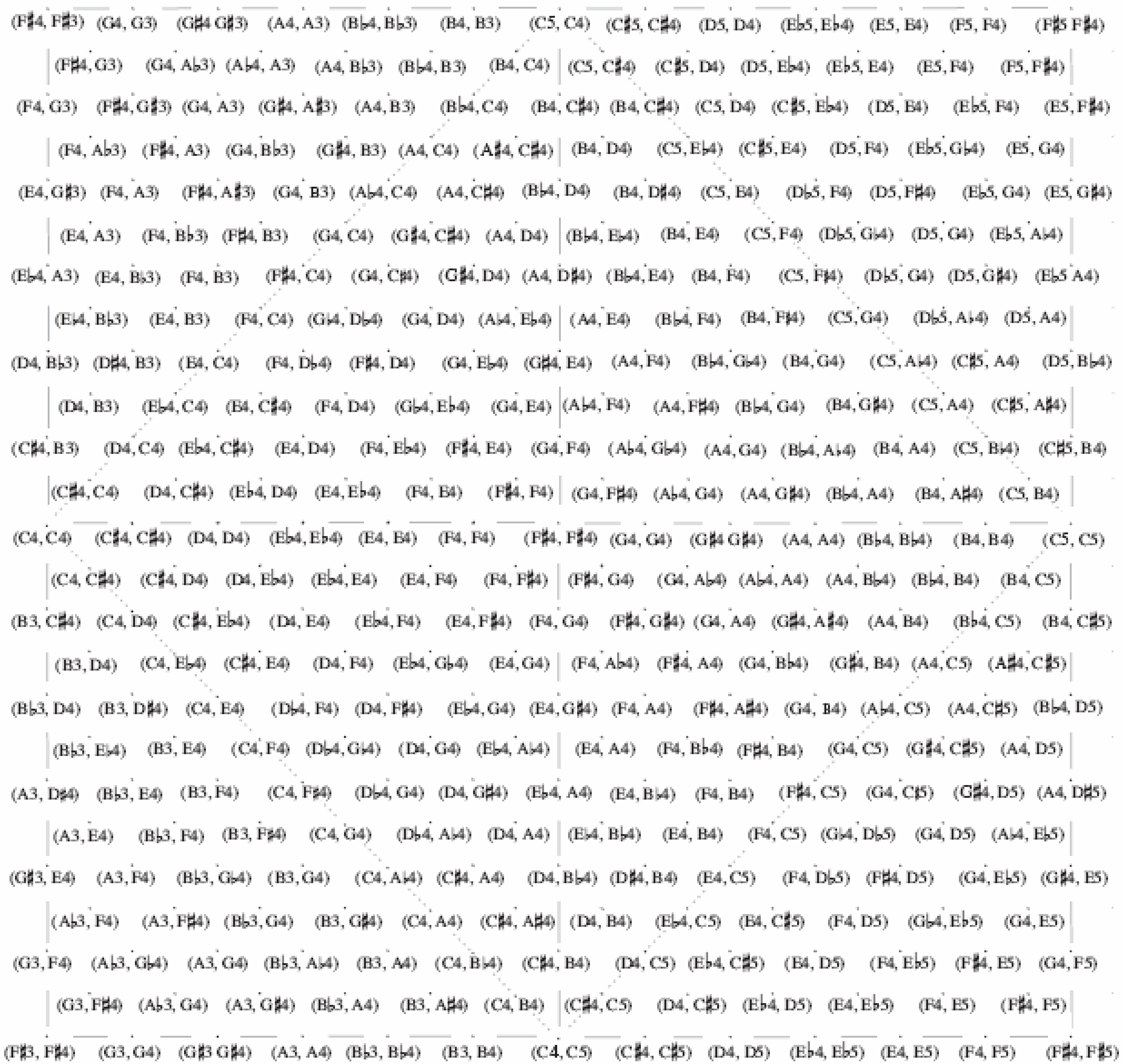


Figure S3. The quotient space $\mathbb{R}/12\mathbb{Z}$ is a circle whose circumference is twelve units long. The twelve familiar pitch classes of Western equal-temperament evenly divide this circle. Since the circle is continuous, it contains a point for every conceivable pitch class. The figure shows the locations of the pitch class 0.17, which is seventeen cents (hundredths of a semitone) above pitch class C, and pitch class 2.5 (D quarter tone sharp), which is halfway between D and E♭.

(a)


(F♯4, F♯3) (G4, G3) (G♯4 G♯3) (A4, A3) (B♭4, B♭3) (B4, B3) (C5, C4) (C♯5, C♯4) (D5, D4) (E♭5, E♭4) (E5, E4) (F5, F4) (F♯5 F♯4)
(F♯4, G3) (G4, A♭3) (A♭4, A3) (A4, B♭3) (B♭4, B3) (B4, C4) (C5, C♯4) (C♯5, D4) (D5, E♭4) (E♭5, E4) (E5, F4) (F5, F♯4)
(F4, G3) (F♯4, G♯3) (G4, A3) (G♯4, A♯3) (A4, B3) (B♭4, C4) (B4, C♯4) (B4, C♯4) (C5, D4) (C♯5, E♭4) (D5, E4) (E♭5, F4) (E5, F♯4)
(F4, A♭3) (F♯4, A3) (G4, B♭3) (G♯4, B3) (A4, C4) (A♯4, C♯4) (B4, D4) (C5, E♭4) (C♯5, E4) (D5, F4) (E♭5, G♭4) (E5, G4)
(E4, G♯3) (F4, A3) (F♯4, A♯3) (G4, B3) (A♭4, C4) (A4, C♯4) (B♭4, D4) (B4, D♯4) (C5, E4) (D♭5, F4) (D5, F♯4) (E♭5, G4) (E5, G♯4)
(E4, A3) (F4, B♭3) (F♯4, B3) (G4, C4) (G♯4, C♯4) (A4, D4) (B♭4, E♭4) (B4, E4) (C5, F4) (D♭5, G♭4) (D5, G4) (E♭5, A♭4)
(E♭4, A3) (E4, B♭3) (F4, B3) (F♯4, C4) (G4, C♯4) (G♯4, D4) (A4, D♯4) (B♭4, E4) (B4, F4) (C5, F♯4) (D♭5, G4) (D5, G♯4) (E♭5 A4)
(E♭4, B♭3) (E4, B3) (F4, C4) (G♭4, D♭4) (G4, D4) (A♭4, E♭4) (A4, E4) (B♭4, F4) (B4, F♯4) (C5, G4) (D♭5, A♭4) (D5, A4)
(D4, B♭3) (D♯4, B3) (E4, C4) (F4, D♭4) (F♯4, D4) (G4, E♭4) (G♯4, E4) (A4, F4) (B♭4, G♭4) (B4, G4) (C5, A♭4) (C♯5, A4) (D5, B♭4)
(D4, B3) (E♭4, C4) (E4, C♯4) (F4, D4) (G♭4, E♭4) (G4, E4) (A♭4, F4) (A4, F♯4) (B♭4, G4) (B4, G♯4) (C5, A4) (C♯5, A♯4)
(C♯4, B3) (D4, C4) (E♭4, C♯4) (E4, D4) (F4, E♭4) (F♯4, E4) (G4, F4) (A♭4, G♭4) (A4, G4) (B♭4, A♭4) (B4, A4) (C5, B♭4) (C♯5, B4)
(C♯4, C4) (D4, C♯4) (E♭4, D4) (E4, E♭4) (F4, E4) (F♯4, F4) (G4, F♯4) (A♭4, G4) (A4, G♯4) (B♭4, A4) (B4, A♯4) (C5, B4)
(C4, C4) (C♯4, C♯4) (D4, D4) (E♭4, E♭4) (E4, E4) (F4, F4) (F♯4, F♯4) (G4, G4) (G♯4 G♯4) (A4, A4) (B♭4, B♭4) (B4, B4) (C5, C5)
(C4, C♯4) (C♯4, D4) (D4, E♭4) (E♭4, E4) (E4, F4) (F4, F♯4) (F♯4, G4) (G4, A♭4) (A♭4, A4) (A4, B♭4) (B♭4, B4) (B4, C5)
(B3, C♯4) (C4, D4) (C♯4, E♭4) (D4, E4) (E♭4, F4) (E4, F♯4) (F4, G4) (F♯4, G♯4) (G4, A4) (G♯4, A♯4) (A4, B4) (B♭4, C5) (B4, C♯5)
(B3, D4) (C4, E♭4) (C♯4, E4) (D4, F4) (E♭4, G♭4) (E4, G4) (F4, A♭4) (F♯4, A4) (G4, B♭4) (G♯4, B4) (A4, C5) (A♯4, C♯5)
(B♭3, D4) (B3, D♯4) (C4, E4) (D♭4, F4) (D4, F♯4) (E♭4, G4) (E4, G♯4) (F4, A4) (F♯4, A♯4) (G4, B4) (A♭4, C5) (A4, C♯5) (B♭4, D5)
(B♭3, E♭4) (B3, E4) (C4, F4) (D♭4, G♭4) (D4, G4) (E♭4, A♭4) (E4, A4) (F4, B♭4) (F♯4, B4) (G4, C5) (G♯4, C♯5) (A4, D5)
(A3, D♯4) (B♭3, E4) (B3, F4) (C4, F♯4) (D♭4, G4) (D4, G♯4) (E♭4, A4) (E4, B♭4) (F4, B4) (F♯4, C5) (G4, C♯5) (G♯4, D5) (A4, D♯5)
(A3, E4) (B♭3, F4) (B3, F♯4) (C4, G4) (D♭4, A♭4) (D4, A4) (E♭4, B♭4) (E4, B4) (F4, C5) (G♭4, D♭5) (G4, D5) (A♭4, E♭5)
(G♯3, E4) (A3, F4) (B♭3, G♭4) (B3, G4) (C4, A♭4) (C♯4, A4) (D4, B♭4) (D♯4, B4) (E4, C5) (F4, D♭5) (F♯4, D5) (G4, E♭5) (G♯4, E5)
(A♭3, F4) (A3, F♯4) (B♭3, G4) (B3, G♯4) (C4, A4) (C♯4, A♯4) (D4, B4) (E♭4, C5) (E4, C♯5) (F4, D5) (G♭4, E♭5) (G4, E5)
(G3, F4) (A♭3, G♭4) (A3, G4) (B♭3, A♭4) (B3, A4) (C4, B♭4) (C♯4, B4) (D4, C5) (E♭4, C♯5) (E4, D5) (F4, E♭5) (F♯4, E5) (G4, F5)
(G3, F♯4) (A♭3, G4) (A3, G♯4) (B♭3, A4) (B3, A♯4) (C4, B4) (C♯4, C5) (D4, C♯5) (E♭4, D5) (E4, E♭5) (F4, E5) (F♯4, F5)
(F♯3, F♯4) (G3, G4) (G♯3 G♯4) (A3, A4) (B♭3, B♭4) (B3, B4) (C4, C5) (C♯4, C♯5) (D4, D5) (E♭4, E♭5) (E4, E5) (F4, F5) (F♯4, F♯5)

(b)

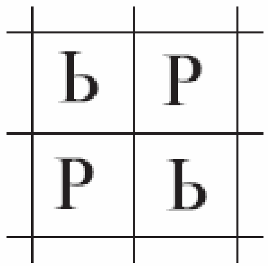


Figure S4. (*a*) A portion of the infinite plane representing ordered two-note chords of pitches. The four quadrants are equivalent to within octave displacement and permutation. Each is equivalent to Figure 2 in the main paper. (*b*) An abstract representation of the symmetries relating the four quadrants. The lower-left quadrant is related to the upper-left by a reflection that preserves their common border. This action permutes each dyad. Translation of any quadrant diagonally up and right transposes the first element of each dyad by an ascending octave. Translation of any quadrant diagonally down and right transposes the second element of each dyad by an ascending octave. These operations suffice to generate the infinite, periodic figure. The space can be described, metaphorically, as wallpaper; Figure 2 in the main paper provides the pattern, Figure S4(*b*) shows how the pattern is to be assembled, and Figure S4(*a*) shows a portion of the result. The diamond in the center of Fig. S4(*a*) corresponds to the 2-torus shown in Figure S9.

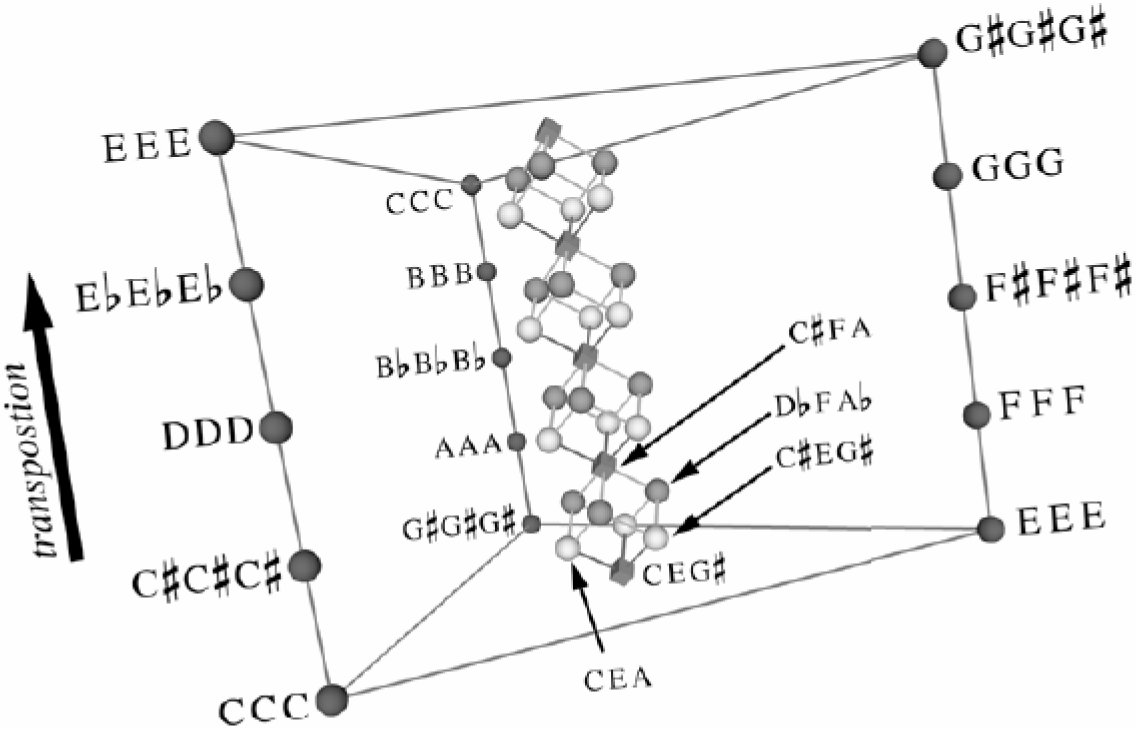
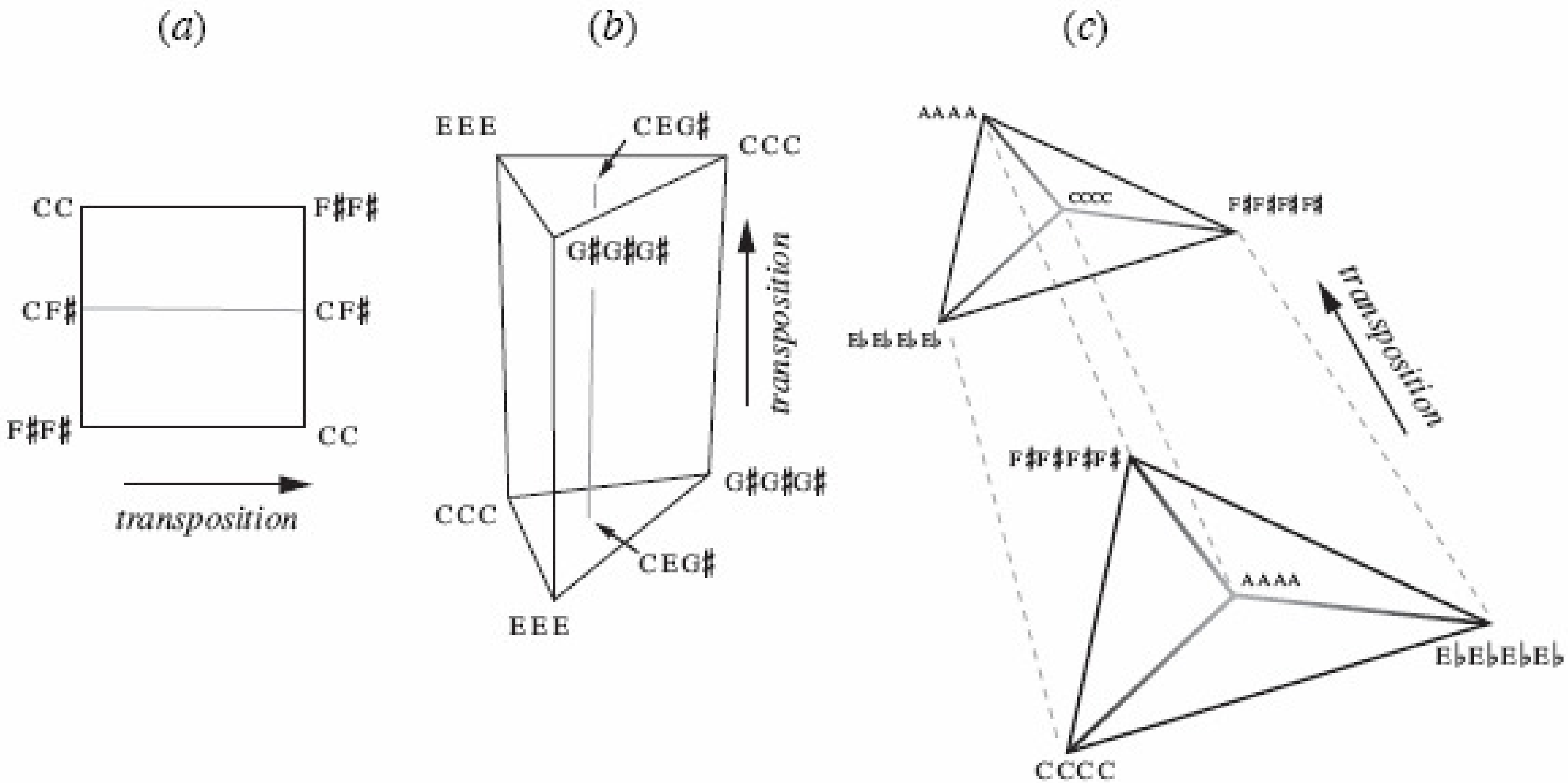


Figure S5. The orbifold $\mathbb{T}^3/\mathcal{S}_3$ is a prism whose two triangular faces are identified by way of a 120° rotation. Several familiar equal-tempered chords are depicted on the figure. Augmented triads, which divide the octave into three equal parts, are shown as dark cubes. Minor chords are light spheres and major chords are dark spheres. Lines connecting augmented, minor, and major chords indicate that they can be linked by voice leading in which a single voice moves by a single semitone. Since minor and major chords divide the octave nearly evenly, they are clustered near the center of the orbifold. Triple unisons, which contain only one pitch class, are found on the edge of the figure.



(a)
CC
F♯F♯
CF♯
CF♯
F♯F♯
CC
transposition
(b)
EEE
CEG♯
CCC
G♯G♯G♯
transposition
CCC
G♯G♯G♯
CEG♯
EEE
(c)
AAAA
CCCC
F♯F♯F♯F♯
E♭E♭E♭E♭
transposition
F♯F♯F♯F♯
AAAA
E♭E♭E♭E♭
CCCC

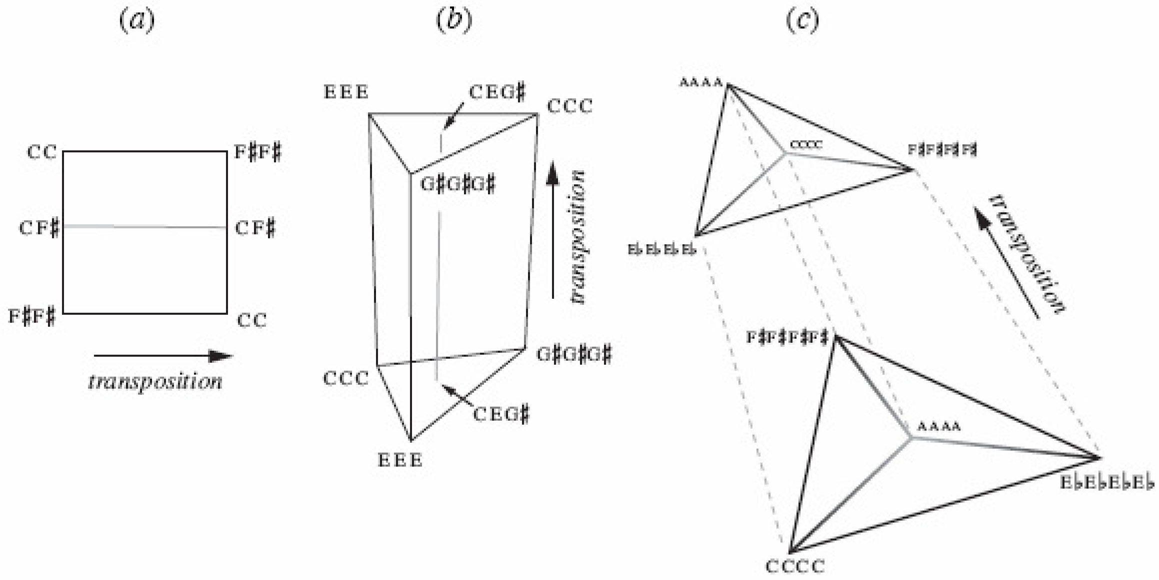


Figure S6. (*a*) The orbifold $\mathbb{T}^2/\mathcal{S}_2$ is a two-dimensional prism (parallelogram) whose base (a line segment) is glued to the opposite face. Before gluing, the base must be twisted so that chords on the left edge match those on the right. This twist is a reflection that can be represented as a rotation in three Euclidean dimensions. The line at the center of the figure contains chords that divide the octave evenly. (*b*) The orbifold $\mathbb{T}^3/\mathcal{S}_3$ is a three-dimensional prism whose two triangular faces are glued together. Before gluing, rotate one face by 120°, so that the chords match. The result is the bounded interior of a twisted triangular 2-torus. Augmented triads, which divide the octave into three equal parts, lie on the line at the center of the figure. Major and minor chords are close to this line, as shown in Figure S5. Rotating the prism around the central line by 120° transposes every chord by major third. (*c*) The orbifold $\mathbb{T}^4/\mathcal{S}_4$ is a four-dimensional prism whose two tetrahedral faces are glued together. The dashed lines extend into the fourth dimension. Before identifying the two faces, twist one so that the chords match. The twist is a reflection, as in the two-dimensional case. Diminished seventh chords, which divide the octave into four equal pieces, lie at the center of the orbifold. Familiar four-note tonal chords lie close to this chord.

Figure S7. (*a*) $|r - p_2| / |r - p_1| = |q_2 - r| / |q_1 - r|$. (*b*) Any line segment that crosses (p_1, q_2) crosses either (p_1, q_1) or (p_2, q_2).

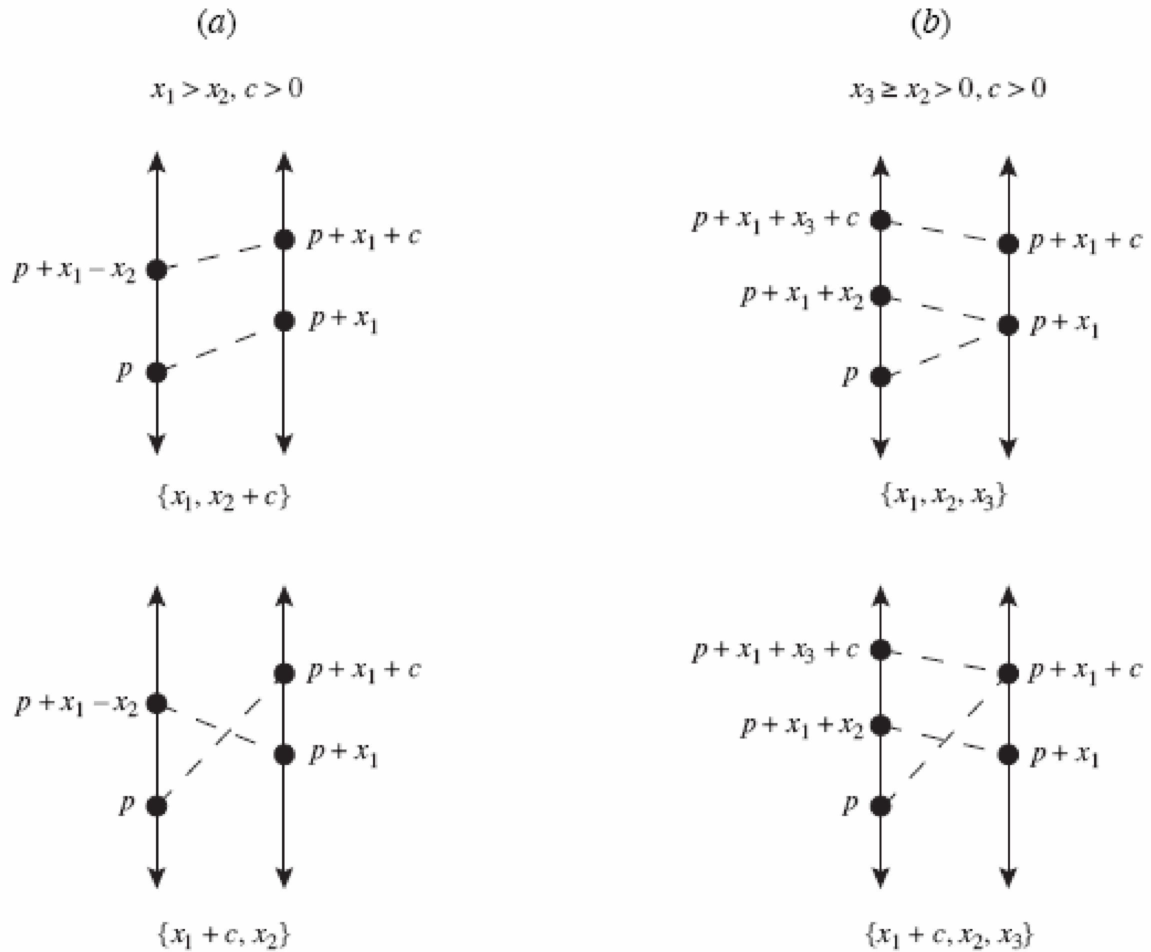


Figure S8. For any violation of the distribution constraint, we can find a crossed voice leading that is smaller than its natural uncrossed alternative. The crossed voice leading at the bottom of each column is smaller than the uncrossed voice leading above it.

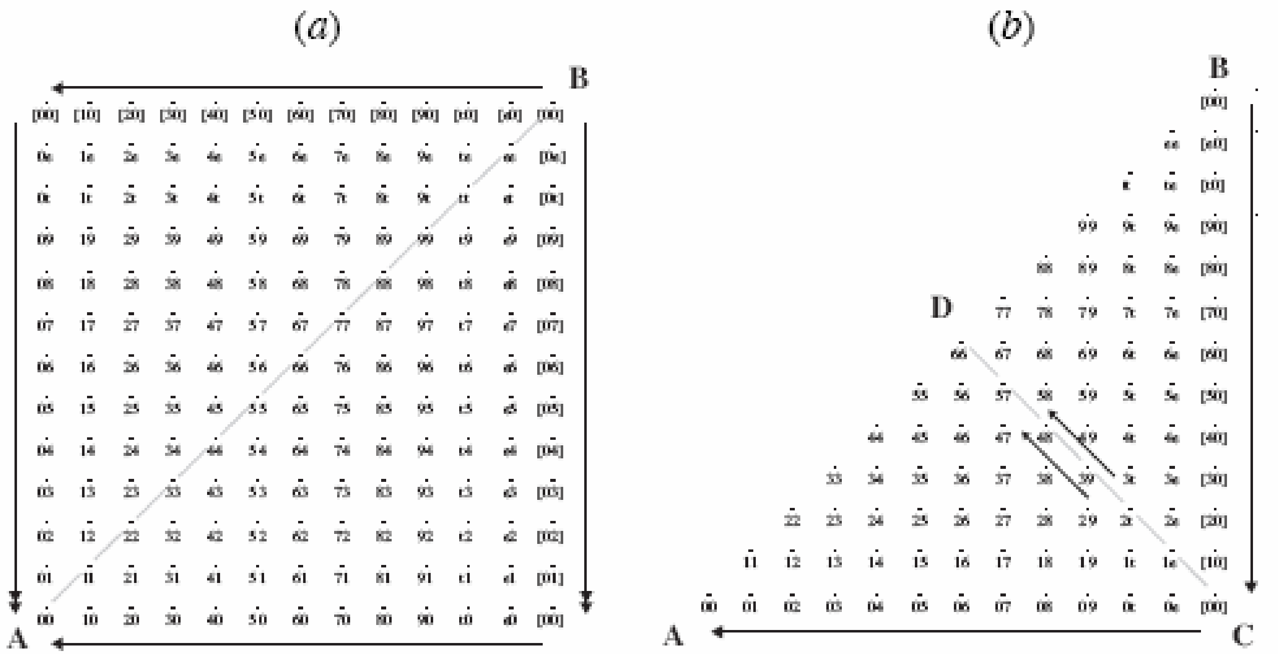


Figure S9. (*a*) The space of ordered two-note chords of pitch-classes is a 2-torus. To identify points (*x*, *y*) and (*y*, *x*), we need to fold the torus along the AB diagonal. The resulting figure, shown in (*b*), is a triangle with two of its sides identified. This is a Möbius strip. To see why, cut figure (*b*) along the line CD and glue AC to CB. (To make this identification in Euclidean 3-space, you will need to turn over one of the pieces of paper.) The result is a square with opposite sides identified, as in Figure 2 of the main paper.

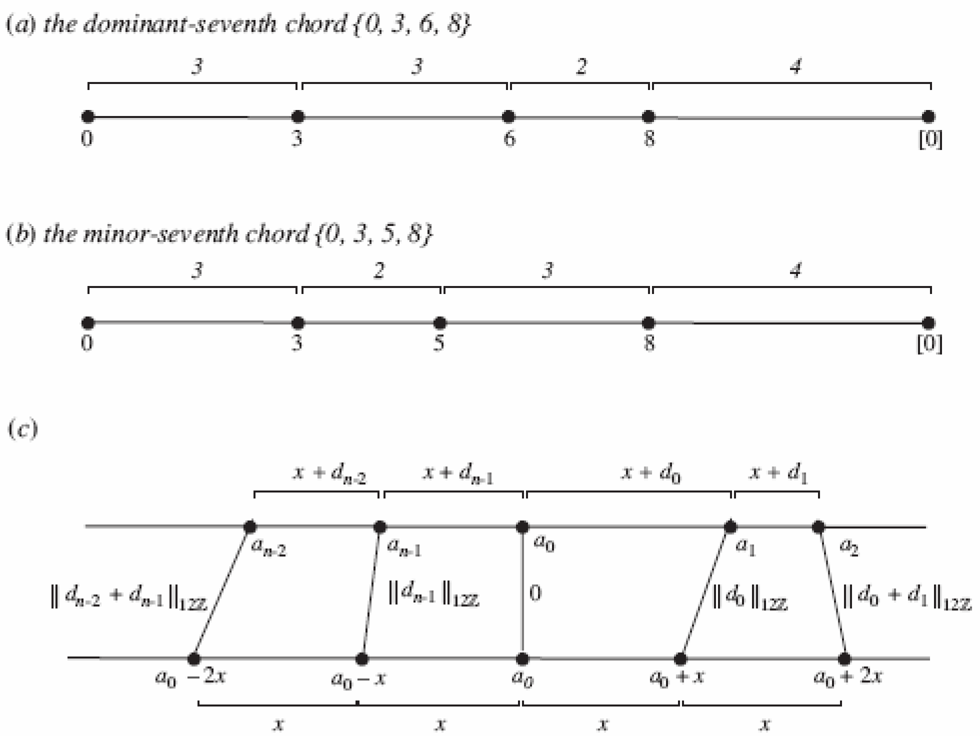


Figure S10. (*a–b*) The dominant-seventh chord {0, 3, 6, 8} and minor-seventh chord {0, 3, 5, 8} can both be arranged as cycles whose notes are 2, 3, 3, and 4 semitones apart. To transform the dominant-seventh chord into a chord that divides the octave evenly, one need only move pitch class 8 to pitch class 9. However, to transform the minor-seventh chord into a chord that divides the octave evenly, one must move at least two pitch classes. (For example, one can move 5 and 8 to 6 and 9, respectively.) Consequently, although both chords have equally small minimal voice leadings to their $\mathbf{T}_3$ forms, the first is closer to the nearest $\mathbf{T}_3$-invariant chord according to many metrics. (*c*) To find a bijective voice leading from any chord to a chord whose intervals are all equal to x, fix a_0, move note a_1 by d_0 semitones, note a_{n-1} by d_{n-1} semitones, note a_2 by $d_0 + d_1$ semitones, and so on. Since pitch-class space is circular, the term $d_{\lfloor n/2 \rfloor}$ will not be involved in this voice leading.

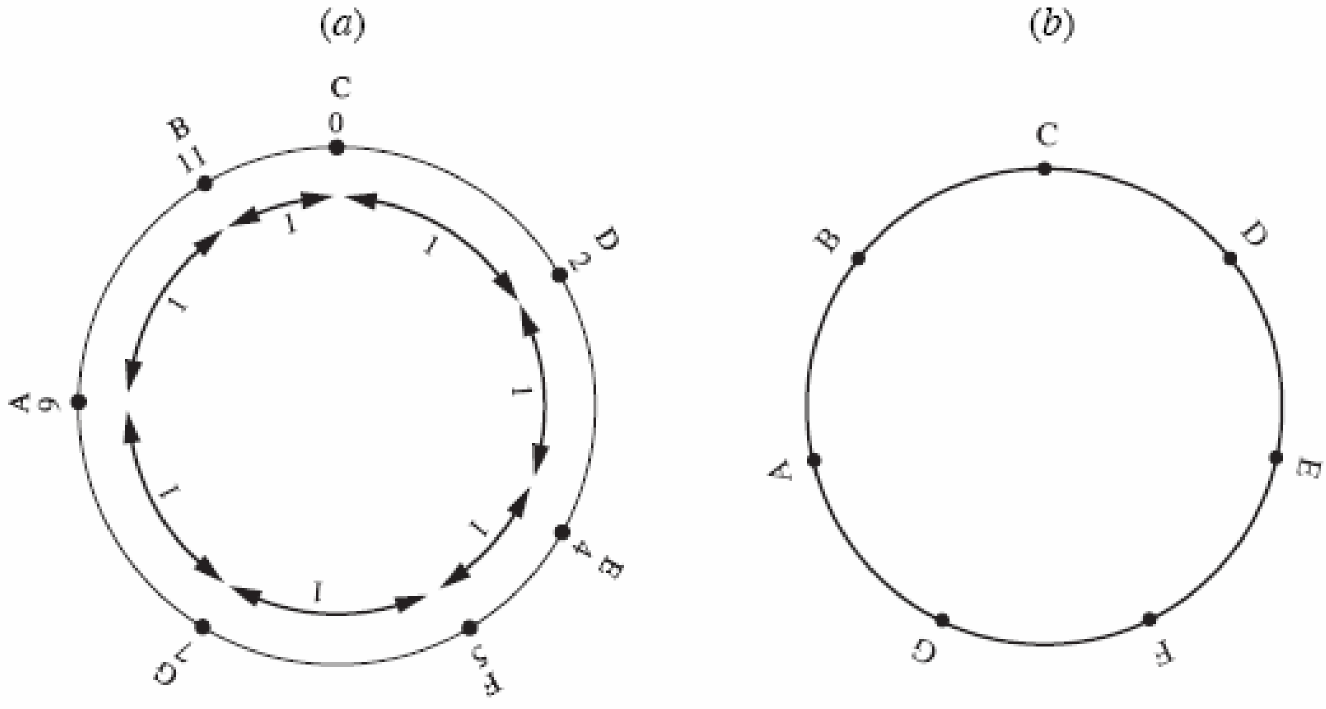


Figure S11. For any scale, we can define a metric such that the scale's notes divide the octave evenly. (*a*) shows the C major scale, as it appears in circular pitch class space. In (*b*), we apply a new metric, so that the scale's notes are equally spaced. The new unit of distance is called a "scale step." Relative to this new metric, the C major triad {C, E, G} and the D minor triad {D, F, A} are related by rotation, which musicians call "scalar transposition."

	4	**8**	**11**	**3**	**4**
4	**(4)→(4)** ***Size: 0***	(4, 4)→(4, 8) *Size: 4*	(4, 4, 4)→ (4, 8, 11) *Size: 9*	(4, 4, 4, 4)→ (4, 8, 11, 3) *Size: 10*	(4, 4, 4, 4, 4)→ (4, 8, 11, 3, 4) *Size: 10*
7	(4, 7)→(4, 4) *Size: 3*	**(4, 7)→(4, 8)** ***Size: 1***	(4, 7, 7)→ (4, 8, 11) *Size: 5*	(4, 7, 7, 7)→ (4, 8, 11, 3) *Size: 9*	(4, 7, 7, 7, 7)→ (4, 8, 11, 3, 4) *Size: 12*
11	(4, 7, 11)→ (4, 4, 4) *Size: 8*	(4, 7, 11)→ (4, 8, 8) *Size: 4*	**(4, 7, 11)→** **(4, 8, 11)** ***Size: 1***	(4, 7, 11, 11)→ (4, 8, 11, 3) *Size: 5*	(4, 7, 11, 11, 11)→ (4, 8, 11, 3, 4) *Size: 10*
0	(4, 7, 11, 0)→ (4, 4, 4, 4) *Size: 12*	(4, 7, 11, 0)→ (4, 8, 8, 8) *Size: 8*	**(4, 7, 11, 0)→** **(4, 8, 11, 11)** ***Size: 2***	(4, 7, 11, 0)→ (4, 8, 11, 3) *Size: 4*	(4, 7, 11, 0, 0)→ (4, 8, 11, 3, 4) *Size: 8*
4	(4, 7, 11, 0, 4)→ (4, 4, 4, 4, 4) *Size: 12*	(4, 7, 11, 0, 4)→ (4, 8, 8, 8, 8) *Size: 12*	(4, 7, 11, 0, 4)→ (4, 8, 11, 11, 11) *Size: 7*	**(4, 7, 11, 0, 4)→** **(4, 8, 11, 11, 3)** ***Size: 3***	**(4, 7, 11, 0, 4, 4)→** **(4, 8, 11, 11, 3, 4)** ***Size: 3***

Figure S12. Using dynamic programming to find a minimal voice leading between {4, 7, 0, 11} and {4, 8, 11, 3}.

2 notes	CG	CF♯
3 notes	CEG	CE♭G♭ CE♭G CEG♯
4 notes	CEGB♭	CE♭G♭A CE♭G♭B♭ CE♭GB♭ CEGB
5 notes	CDEGB♭	CDEGA CDEGB
6 notes	CDEF♯GB♭	CDE♭FGB♭ CDEF♯G♯B♭
7 notes	CDEF♯GAB♭	CDEFGAB♭ CDE♭F♯GAB♭

Dies war die erste Veröffentlichung
in Science, die sich mit Musik beschäftigt:

Science **313**, 2006, 72-74

Time, February 5, 2007
www.music.princeton.edu/dmitri

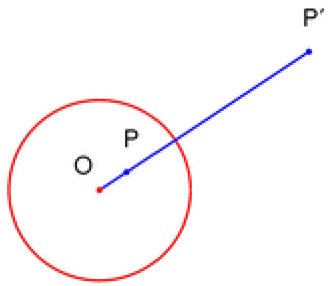
Translational symmetry

Translational symmetry leaves an object invariant under a discrete or continuous group of translations $T_a(p) = p + a$

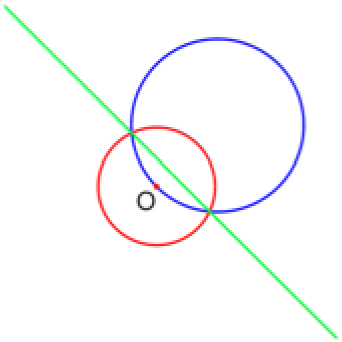
from wikipedia

Circle inversion

$$OP \times OP' = R^2.$$

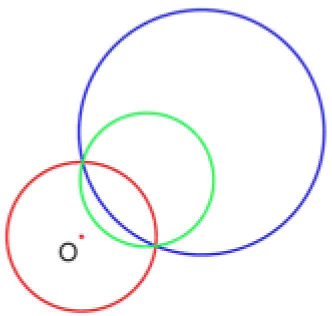


P ' is the inverse of *P* with respect to the circle.
In the plane, the **inverse** of a point *P* in respect to a circle of center *O* and radius *R* is a point *P'* such that *P* and *P'* are on the same ray going from *O*, and *OP* times *OP'* equals the radius squared,



This circle in respect to which inversion is performed will be called the *reference circle*.

The inverse in respect to the red circle of a circle going through *O* (blue), is a line not going through *O* (green), and vice-versa.



The inverse in respect to the red circle of a circle *not* going through *O* (blue), is a circle not going through *O* (green), and vice-versa.

from wikipedia

Reflection formula

In mathematics, a **reflection formula** or **reflection relation** for a function *f* is a relationship between $f(a-x)$ and $f(x)$. It is a special case of a functional equation, and it is very common in the literature to refer to use the term "functional equation" when "reflection formula" is meant.

The even and odd functions satisfy simple reflection relations around $a=0$. For all even functions,

$$f(-x) = f(x),$$

and for all odd functions,

$$f(-x) = -f(x).$$

from wikipedia

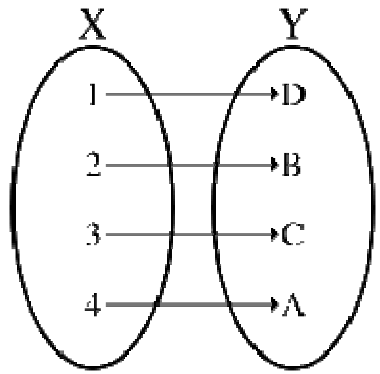
A bijective function.

In mathematics, a function *f* from a set *X* to a set *Y* is said to be **bijective** if for every *y* in *Y* there is exactly one *x* in *X* such that $f(x) = y$.

Said another way, *f* is bijective if it is a **one-to-one correspondence** between those sets; i.e., both **one-to-one** (injective) and **onto** (surjective).

For example, consider the function succ, defined from the set of integers $\mathbb{Z}$ to $\mathbb{Z}$, that to each integer *x* associates the integer succ(*x*) = x + 1. For another example, consider the function sumdif that to each pair (x,y) of real numbers associates the pair sumdif$(x,y) = (x + y, x - y)$.

A bijective function is also called a **bijection** or **permutation**. The latter is more commonly used when $X = Y$. It should be noted that *one-to-one function* means *one-to-one correspondence* (i.e., *bijection*) to some authors, but *injection* to others. The set of all bijections from *X* to *Y* is denoted as $X \leftrightarrow Y$.



from wikipedia

Das Thema der überlagerten offensichtlichen und verborgenen Symmetrien am Beispiel des Schachbretts wurde im Vortrag nur mündlich erwähnt und dieser Präsentation nachträglich hinzugefügt. Dies gilt auch für den Spruch von Adolphe Quetelet.

14.3.2007 Jutta Köhler